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FIXED POINT THEOREMS IN BANACH SPACE USING NOOR ITERATION

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ABSTRACT

In this paper introduces a new idea for fixed point of nonexpansive mapping using Noor iterative schemes and we prove fixed point theorems for weakly convergence of sequence in Banach space which satisfies Opial's condition MSC :47H09, 47H10

KEYWORDS: Noor, nonexpansive mapping mapping, weakly convergence

Let E be a closed bounded convex subset of a Banach space X and T: E→E be a mapping. Then T is known as nonexpansive if.

$$\|T(x) - T(y)\| \quad \|x - y\| \qquad \forall x, y \in E. \qquad (1.1)$$

Let $F(T) = \{x \;\; E : T(x) = x\}$, then F(T) is called the set of fixed points of mapping T. If E is a closed and convex subset of a Hilbert space H and T has a fixed point, then for every $x \in E$, $\{T_n(x)\}$ is weakly almost convergent to a fixed point of T, as $n \to \propto$. This theorem is called the first ergodic theorem which was proved by Baillon (1975) for general nonexpansive mappings in Hilbert space H.

Diaz and Metcalf (1969) studied quasi-nonexpansive mapping in Banach space. This type of work was also given by some Mathematician Awasthy and Shukla (2008); Ghosh and Debnath (1997), Kirk (1997); Opial (1967), Owojori and Olessegun (2006); Pazy (1977), Rhoades and Temir (2006); Shahzad (2004), Xu and Noor (2007) and Zhou et al.(2002). Recently, this concept was given by Kirk (1997) in metric spaces which we adapt to a normed spaces. The mapping T is known as quasi-nonexpansive mapping if

$$\|Tx - f\| \leq \|x - f\| \qquad (1.2)$$

for all $x \in E$ and $f \in F(T)$.

Preliminaries and Definitions

Let X be a Banach space and T: E→E be a mapping of a convex subset of X. Let $x_1 \in E$. A sequence $\{x_n\}$ is obtained by $x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, Ty_n$

with $\quad y_n = (1 - \beta_n)\, x_n + \beta_n\, Tz_n$

$$z_n = (1 - \gamma_n)\, x_n + \gamma_n\, Tx_n \qquad n \geq 1 \qquad (2.1)$$

and with $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfying following conditions

(i) $\quad 0 \leq \alpha_n \leq \beta_n \leq \gamma_n < 1$

(ii) $\quad \lim_{n\to\infty} \beta_n = 0, \quad \lim_{n\to\infty} \gamma_n = 0, \qquad (2.2)$

(iii) $\quad \sum_{n=1}^{\infty} \beta_n \gamma_n = \infty$

The iterates given in (2.1) with (2.2) are known as Noor iteration. Noor iteration can be written as follows:

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, T[(1 - \beta_n)\, x_n + \beta_n T\{(1 - \gamma_n)\, x_n + \gamma_n Tx_n\}] \quad (2.3)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1)

A Banach space X is said to satisfy Opial's condition Noor (2000) if for each sequence $\{x_n\}$ in X such that $x_n \to x$ implies that

$$\overline{\lim_{n\to\infty}} \|x_n - x\| < \overline{\lim_{n\to\infty}} \|x_n - y\|$$

for all $y \in x$ with $y \neq x$.

In this paper, we consider that T is a nonexpansive mapping in the Banach space X. Then we will show the weak convergence of the sequence of Noor iterates to a fixed point of T.

RESULTS

Theorem: 3.1

The E be closed convex bounded subset of uniformly convex Banach space X which satisfies Opial's condition. Let T be a self mapping of E and T is a

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nonexpansive mapping on E. Then for $x_0 \in E$, the sequence $\{x_n\}$ of Noor iterates converges weakly to fixed point of F(T).

Proof

If (F(T) is nonempty and a singleton then the proof is complete. Let us consider that F(T) is nonempty and F(T) is not a singleton. Then

$$||x_{n+1} - f||$$
$$=||(1-\alpha_n)x_n+\alpha_n T[(1-\beta_n)x_n+\beta_n T\{(1-\gamma_n)x_n+\gamma_n Tx_n\}]-(1-\alpha_n+\alpha_n) f||$$
$$=||(1-\alpha_n)(x_n-f)+\alpha_n [T[(1-\beta_n)x_n+\beta_n T\{(1-\gamma_n)x_n+\gamma_n Tx_n\}] - f]||$$
$$\leq||(1-\alpha_n)(x_n-f)|| + ||\alpha_n [T[(1-\beta_n)x_n+\beta_n T\{(1-\gamma_n)x_n+\gamma_n Tx_n\}] - f||$$
$$\leq(1-\alpha_n)||(x_n-f)||+\alpha_n ||T[(1-\beta_n)x_n+\beta_n T\{(1-\gamma_n)x_n+\gamma_n Tx_n\}] - f ||$$
$$\leq(1-\alpha_n)||x_n-f||+\alpha_n||T[(1-\beta_n)x_n+\beta_n T(1-\gamma_n)x_n+_n Tx_n\}]-(1-\gamma_n+_n)f||$$
$$\leq(1-\alpha_n)||x_n-f||+\alpha_n||(1-\beta_n)x_n+\beta_n T\{(1-\gamma_n)x_n+\gamma_n Tx_n\}-(1-\gamma_n)f+\gamma_n f||$$
$$\leq(1-\alpha_n)||x_n-f||+\alpha_n||(1-\beta_n)(x_n-f)+\beta_n T\{(1-\gamma_n)x_n+\gamma_n Tx_n\}-\gamma_n f||$$
$$\leq(1-\alpha_n)||x_n-f||+\alpha_n||(1-\beta_n)(x_n-f)+\beta_n\{(1-_n)x_n+\gamma_n Tx_n\}-\gamma_n f||$$
$$\leq(1-\alpha_n)||x_n-f||+\alpha_n||(1-\beta_n)(x_n-f)+\beta_n\{(1-\gamma_n)x_n+\gamma_n Tx_n\}-(1-\gamma_n+\gamma_n)_n f||$$
$$\leq(1-\alpha_n)||x_n-f||+\alpha_n||(1-\beta_n)(x_n-f)+\beta_n(1-\gamma_n)(x_n-f)+\beta_n\gamma_n Tx_n-\beta_n\gamma_n f||$$
$$\leq(1-\alpha_n)||x_n-f||+\alpha_n||(1-\beta_n)(x_n-f)+\beta_n(1-\gamma_n)(x_n-f)+\beta_n\gamma_n x_n-\beta_n\gamma_n f||$$
$$\leq(1-\alpha_n)||x_n-f||+\alpha_n||(1-\beta_n)(x_n-f)+\beta_n(1-\gamma_n)(x_n-f)+\beta_n\gamma_n(x_n-f)||$$
$$\leq(1-\alpha_n)||x_n-f||+\alpha_n||x_n-f||$$
$$=||x_n-f||$$
$$||x_{n+1}-f||\leq||x_n-f||$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1)

Hence for $\alpha_n\neq0,\beta_n\neq0,\gamma_n\neq0\{||x_n-f||\}$ is a non increasing sequence. Then $\lim_{n\to\infty}||x_n-f||$ exists.

Now we show that $\{x_n\}$ converges to a fixed point of T. The sequence $\{x_n\}$ contains a subsequence which converges weakly to a point in E. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be two subsequences of $\{x_n\}$ which converges weakly to f and q respectively. We will show that f = q.

Suppose that X satisfies Opial's condition and that $f \neq q$ is in a weak limit set of the sequence $\{x_n\}$ Then $\{x_{n_k}\} \to f$ and $\{x_{m_k}\} \to q$ respectively. Since $\lim_{n\to\infty}||x_n-f||$ exist for any $f \in F(T)$.

By Opial's condition, we conclude that

$$\lim_{n\to\infty}||x_n-f|| = \lim_{k\to\infty}||x_{n_k}-f|| < \lim_{k\to\infty}||x_{n_k}-q||$$
$$= \lim_{n\to\infty}||x_n-q|| = \lim_{j\to\infty}||x_{n_j}-q||$$
$$< \lim_{j\to\infty}||x_{m_j}-f|| = \lim_{n\to\infty}||x_n-f||$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to an element of F(T).

Theorem: 3.2

Let E be a closed convex bounded subset of uniformly convex Banach space X, which satisfies Opial's condition. Let T be a self mapping of E and T is a quasi-nonexpansive mapping on E. Then for $x_0 \in E$, the sequence $\{x_n\}$ of Noor iterates converges weakly to fixed point of F(T).

Proof

Since every nonexpansive mapping is a quasi-nonexpansive mapping. Proof of this theorem is similar to above theorem.

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